Final - Optimization (2020-21)
Time: 3.5 hours.
Attempt all questions. The total marks is 50.
You may quote any result proved in class without proof.

1. Consider the problem: minimize $2|x_1| + 5|x_2 - 6|$ subject to $|x_1 + 3| + x_2 \geq 3$. Reformulate this as a linear programming problem. **[4 marks]**

2. Give an example of a linear program $\min_{\mathbf{x} \in P} \mathbf{c}^T \mathbf{x}$, where the polyhedron P has an extreme point and the linear program has optimal cost equal to $-\infty$. **[3 marks]**

3. A polyhedron is represented by a system of equality and inequality constraints.
   (a) Give an example of a polyhedron P and a point $\mathbf{x}$, and two representations of P, such that $\mathbf{x}$ is a basic solution under one representation but is not a basic solution under the other representation. **[2 marks]**
   (b) Can one create an example of P with two representations, and an $\mathbf{x}$ such that $\mathbf{x}$ is a basic *feasible* solution (bfs) under one representation and not a bfs under the other representation? Explain. **[2 marks]**

4. Consider the following polyhedron in three dimensions: $x + 2y + 3z \geq 5,\ 2x + y + z \geq 6$. Find a line that is contained in the polyhedron. **[4 marks]**

5. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a convex function and let $S \subset \mathbf{R}^n$ be a convex set. Let $\mathbf{x}^*$ be an element of S. Suppose that $\mathbf{x}^*$ is a local optimum for the problem of minimizing $f(\mathbf{x})$ over S; that is, there exists some $\epsilon > 0$ such that $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$ for which $\|\mathbf{x} - \mathbf{x}^*\| \leq \epsilon$. Prove that $\mathbf{x}^*$ is globally optimal; that is, $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$. **[4 marks]**

6. Consider the following polyhedron in standard form: $2x_1 + x_2 + 2x_3 + x_4 = 3$, $x_1 + x_2 + 5x_3 + 3x_4 = 2$ and all $x_1, x_2, x_3, x_4 \geq 0$.
   (a) Find the basic feasible solution where x_1, x_2 are the basic variables. **[1 mark]**
   (b) Find the 3rd basic direction. **[2 marks]**
   (c) If we are minimizing $x_1 + x_2 + x_3 + x_4$ over the polyhedron, what is the reduced cost along the 3rd basic direction? **[2 marks]**

7. Let $\mathbf{A}$ be a square matrix with positive entries, and suppose the sum of the entries in each row is equal to 1. Show that 1 is an eigenvalue and that every other eigenvalue λ satisfies $|\lambda| < 1$. **[5 marks]**

8. Solve the following linear program. **[5 marks]**

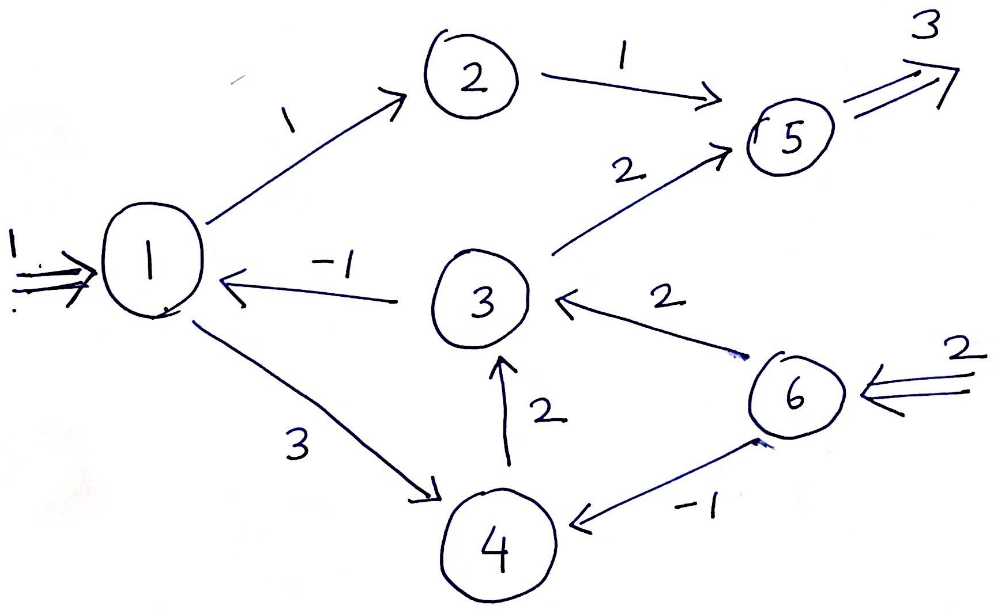
$$\begin{array}{rrrrl} \text{minimize} & -3x_1 & -2x_2 & +5x_3 & \\ \text{such that} & 4x_1 & -2x_2 & +2x_3 & \leq 4, \\ & 2x_1 & -x_2 & +x_3 & \leq 1, \\ \text{and all} & x_1, & x_2, & x_3 & \geq 0. \end{array}$$

9. Consider the following linear program: minimize $-x_1 - 2x_2 + x_3 + 2x_4 - 6x_5$ subject to $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \leq 0,\ x_4 \leq 0$ and x_5 free, and

$$\begin{array}{rrrrrl} -x_1 & & -x_3 & +x_4 & -x_5 & \geq 1 \\ & -x_2 & +x_3 & +4x_4 & -2x_5 & \geq 3. \end{array}$$

   (a) Find the dual of the above problem. **[3 marks]**
   (b) Find the optimal cost of the above problem. **[4 marks]**

10. Consider the *uncapacitated* network flow problem on the directed graph shown below. The numbers next to each directed arc $\rightarrow$ is the *cost* associated to the arc, while the numbers next to $\Rightarrow$ is the external *supply/demand* at the node.



Denote by $\mathbf{c}$ the vector of costs corresponding to the arc. We are interested in minimizing the total cost $\mathbf{c}^T\mathbf{f}$, where the flow vector $\mathbf{f}$ satisfies the flow conservation equations and $\mathbf{f} \geq \mathbf{0}$.

(a) Find an optimal basic feasible solution (feasible tree solution) to the problem. **[7 marks]**

(b) Find the optimal cost for the problem. **[2 marks]**